THE THEORY OF SHADOW RAILS

BY W. H. JACKSON

1. Introduction. IT is a common experience to view two sets of railings superposed: when this occurs there are generally to be observed darker vertical bands forming *shadow* rails on a larger scale. The theory of the formation of these bands is the subject of the present paper.

Perhaps, however, the chief interest of the investigation lies in the fact that the discussion enables us to form a very simple model of the phenomena of *group velocity*, observed whenever one train of waves is superposed upon another train of different wave length travelling with a different velocity. The characteristic equation, (20), is obtained in Section 6.

The theory will be simplified by supposing that the rails have no appreciable thickness, only length and breadth, and that they are uniformly spaced, all the rails of one set being in the same plane.

There are no figures illustrating the present paper because it is so easy actually to produce the phenomena described and to do this is so much more useful than to rely upon drawings. Cut out alternate rows of squares from two superposed sheets of squared paper, cutting out less rather than more. If the sheets be separated and held some little distance apart and a dark ground be viewed through the slits made as described, all the phenomena to be discussed can readily be observed.

Let us suppose that the two sets of uniform rails are in parallel planes. Those in the farther plane may be replaced, so far as this phenomenon is concerned, by uniformly spaced rails in the nearer plane. In a given space L let there be n rails of the first set, n' rails of the second set and N shadow rails. By analogy with the theory of beats in sound, we see at once that

$$N = n' - n. \tag{1}$$

For suppose that $n' > n$, so that the distance between the centres of two consecutive rails of the first set is greater than that between two consecutive rails of the second set. The shadow spaces correspond to places where the two sets of rails have their centers nearly coincident and the shadow rails correspond to places where the rails of one set block the spaces of the other set.

Thus in the distance between the centres of two consecutive shadow spaces, there will be just one more of the second set of rails than of the first set. If in any distance large compared with the distance between consecutive shadow spaces there are N shadow rails, this must be the excess of rails of the second set over rails of the first set. But this is equation (1) put into words.

This intuitive result leads us to expect the average spacing of the shadow rails to be entirely independent of the breadths of the two sets of rails: a result which a more detailed examination confirms.

The following phenomena may easily be observed. As the observer walks past two parallel sets of rails of the same size the shadows travel along with him. If the two sets are not both vertical the slope of the more distant set relative to the nearer set will be exaggerated. But if the nearer set is much more closely spaced than the other both these results are reversed.

If two sets of rails meet at a corner, the shadow rails appear to converge towards or diverge from one special spot according to the direction of motion of the observer.

Finally, if the shadows of a set of rails cast on level ground by the sun be viewed through the rails the shadow bands are found to be curved instead of straight.

The only reference to the subject which the writer has seen is an example given in Chrystal's Algebra,* where the term *ghosts* is applied to places at which the rails appear crowded together. But in that case the rails do not overlap and the shadow rails here considered would not be visible.

Although the theory of the present paper is limited to that case in which uniformly spaced rails produce uniformly spaced shadow rails it is not difficult to trace roughly what would be the changes introduced if the cross-section of the rails instead of being a mathematical line were a rectangle or a circle. In these cases as the eye travels outwards towards the more distant portions of the railing, a point will soon be reached at which it becomes totally opaque. If the rails be replaced by equivalent flat rails, the breadth of these equivalent rails will increase as the eye travels outwards, until the spaces entirely disappear.

It was the study of the phenomena here discussed which led to the theory of integral multiples whose residues, mod 1, lie within given limits, which forms the subject of the preceding paper.

---

* Part II, Chapter XXXII, Exercises XXX, no. 7.

2. Notation. Let $2a$ be the distance between the centers of consecutive rails of the first set and let $2b$ be the breadth of each rail. Let $2a'$, $2b'$ denote corresponding magnitudes for the second set, when viewed in the plane of the first, and suppose that a is greater than a'.

The condition that any space of the first set is entirely closed by a rail of the second set is that c, the distance between their centers, must be less than $b' - (a - b)$. Two cases therefore arise according as b' is greater or less than $a - b$. Only in the former case is it possible for shadow rails to be observed. In the second case, nothing but ghosts, places which are darker because the spaces between consecutive rails are narrowed but not obscured, will be visible.

Let c_r denote the distance of the rth rail-center of the second set from the nearest space-center of the first set and let the direction in which the counting proceeds be the positive direction. Then

$$c_{r+1} = c_r - 2(a - a'), \qquad \text{provided that} \qquad |c_{r+1}| < a. \tag{2}$$

The co-existence of two consecutive values of c, each numerically less than a', implies that

$$a - a' < a' \qquad \text{and therefore} \qquad a' > \tfrac{1}{2}a.^* \tag{3}$$

More generally, the problem to be solved is to find for what values of n, n' the following inequalities are satisfied.

$$b' + b - a \geqq c + 2n'a' - 2na \geqq -(b' + b - a), \tag{4}$$

where $a' > c > -a'$, $a > b$, $a > a' > b'$.

The values of n satisfying (4) determine the numbers of the spaces of the first set which are completely blocked by rails of the second set.

If we write now

$$2a'\beta' = c - (b' + b - a), \quad 2a'\beta = c + (b' + b - a), \; d = a/a', \tag{5}$$

inequalities (4) may be written

$$n' + \beta \geqq nd \geqq n' + \beta', \tag{6}$$

n, n' being integers, and the problem is therefore reduced to that discussed in the preceding paper.

---

* If $\frac{1}{3}a < a' < \frac{1}{2}a$, we should write $c_{r+1} = c_r - 2(a - 2a')$.

But in that paper a detailed investigation was made whereas now it is only the simpler average phenomena which come under discussion.

The general condition that shadow rails should exist can readily be written down as a consequence of the result proved at the end of the preceding paper (p. 140). In the notation there used we must find the condition that $L = 1$. A *sufficient* condition is that

$$\beta - \beta' = \frac{b' + b - a}{a'} > w_1 \cdot \tag{7}$$

But this is not necessary if $q_2 = a_2 = 1$. In this case the required necessary and sufficient condition is that

$$b' + b - a > a'w_2. \tag{8}$$

3. The breadth of the shadow rails. The boundaries of a shadow rail are determined by rails of the second set which we shall distinguish by the numbers 0 and n, and which satisfy the following inequalities.

$$c_0 > b' + b - a \geqq c_1 > c_2 \cdot \cdot \cdot > c_{n-1} \geqq -(b' + b - a) > c_n. \tag{9}$$

THEOREM I. *The boundaries of the shadow rails are formed by rails of the more closely spaced set.*

It follows from (9) that c_0 is positive, c_n negative. But c_0, c_n refers to rails of the second set which just fail to cover the nearest space of the first set and hence the open spaces occur on the negative and positive sides in the two cases respectively. It was previously postulated that the direction of counting should be the positive direction and therefore the open spaces which adjoin the rails denoted by the suffixes 0, n lie on the side *away* from the shadow rail. The boundaries of the shadow rail are therefore the boundaries of these two rails which belong to the second set.

THEOREM II. *The breadth of any shadow rail differs from*

$$2\,\frac{b/a + b/a' - a'/a}{1/a' - 1/a}$$

by less than $2a'$.

It follows from inequalities (9) and equations (2) that

$$c_0 > b' + b - a \geqq c_0 - 2(a - a'),$$

$$c_0 - 2(n-1)(a - a') \geqq -(b' + b - a) > c_0 - 2n(a - a').$$

From this it follows that

$$2(a - a') \geqq c_0 - (b' + b - a) > 0, \tag{10}$$

$$2(a - a') \geqq 2n(a - a') - c_0 - (b' + b - a) > 0. \tag{11}$$

Therefore $n - 1$ is the greatest integer contained in

$$\frac{c_0 + b' + b - a}{2(a - a')}$$

and, by addition of the two sets of inequalities (10) and (11), n never differs from

$$\frac{b' + b - a}{a - a'} + 1, = \frac{b' + b - a'}{a - a'},$$

by more than unity.

But the breadth of the shadow rail is $2na' + 2b'$, which therefore differs from

$$2\,\frac{a'b + ab' - a'^2}{a - a'}$$

by less than $2a'$, and this is equivalent to the statement made above.

Corollary. It can be shown by the method of the preceding paper that the average breadth of a large number of consecutive shadow rails approaches $2B$ as a limit, where

$$B = \frac{b/a + b'/a' - a'/a}{1/a' - 1/a}. \tag{12}$$

4. The spacing of the shadow rails.

THEOREM III. *The average space between the corresponding boundaries of a large number of consecutive shadow rails approaches $2A$ as a limit, where*

$$\frac{1}{A} = \frac{1}{a'} - \frac{1}{a}.$$

With reference to the shadow rail next to the one already considered we may write, corresponding to inequalities (10),

$$2(a - a') \geqq c_0 - 2r(a - a') + 2a - (b' + b - a) > 0. \tag{13}$$

From (13) it follows that r is the greatest integer less than

$$\frac{c_0 + 2a - (b' + b - a)}{2(a - a')}$$

and from (10) and (13) together it follows that

$$2(a - a') > 2a - 2r(a - a') > -2(a - a'). \tag{14}$$

Similarly, if the first rail in the Nth shadow rail after the one first considered is denoted by the suffix r_N, we have corresponding to inequalities (14)

$$1 > \frac{Na}{a - a'} - r_N > -1,$$

from which it follows that

$$\lim_{N=\infty} \left(\frac{r_N}{N}\right) = \frac{a}{a - a'}.$$

Therefore

$$A = \lim_{N=\infty} \left(\frac{r_N a'}{N}\right) = \frac{aa'}{a - a'},$$

which is equivalent to the result stated above.

Corollary. If n, n', N have the meanings used in equation (1), Theorem III may be stated in the form

$$\lim_{N=\infty} \left(\frac{n' - n}{N}\right) = 1. \tag{15}$$

Similarly the corollary to Theorem II may be stated in the form

$$\lim_{N=\infty} \left(\frac{2nb + 2n'b' - 2NB}{L}\right) = \lim_{N=\infty} \left(\frac{n}{n'}\right). \tag{16}$$

That is, in any given distance the total space covered by the rails of the two sets taken separately exceeds the space covered by the shadow rails by an amount which depends only on n, n' and *is independent of the breadths* of the two sets of rails.

5. Ghosts. If we seek to extend the results of the two last sections to ghosts, we must no longer compare the *beginnings* of two consecutive bands because in this case the shadow has no definite limits. But we can

pick out in each ghost one rail of the second set for which

$$\tfrac{1}{2}(a - a') \geqq c > -\tfrac{1}{2}(a - a') \tag{17}$$

and call this the center rail.

THEOREM IV. *The average space between center rails of a large number of consecutive ghosts approaches $2A$ as a limit, where*

$$\frac{1}{A} = \frac{1}{a'} - \frac{1}{a}.$$

With reference to the Nth ghost beyond the one already considered

$$\tfrac{1}{2}(a - a') \geqq c - 2r_N(a - a') + 2Na > -\tfrac{1}{2}(a - a'). \tag{18}$$

From inequalities (17) and (18)

$$(a - a') > 2Na - 2r_N(a - a') > -(a - a'),$$

whence it follows that

$$\frac{1}{2} > \frac{Na}{a - a'} - r_N > -\frac{1}{2}.$$

From this point the proof proceeds exactly as in Theorem III.

6. The effect of motion. The edge of a shadow rail always coincides with that of a rail of the second set. Hence if the second set moves with velocity v relative to the first set, the relative velocity of the edge of the shadow rail is in general equal to v but an instantaneous jump occurs whenever this shadow edge changes from one rail to the next resulting in a displacement $2a'$. The space-time graph for the shadow edge is therefore made up of a series of steps. First a distance $2(a - a')$ at uniform velocity v, next an instantaneous jump of $2a'$, giving an average velocity V relative to the first set where

$$V = \frac{av}{a - a'}. \tag{19}$$

In the application to group velocity suppose two sets of waves superposed of wave lengths λ, $\lambda + \delta\lambda$ and travelling with velocities v, $v + \delta v$ respectively. It follows from (19) that the group velocity is

$$V = v - \lambda \frac{dv}{d\lambda}. \tag{20}$$

Suppose an observer to walk with velocity v_0 parallel to two sets of rails and at a distance from them of d and d' respectively. Let $2a$, $2a'$ be the distances between the centers of consecutive rails in the two sets respectively. Viewed in the plane of the nearer set, the second set are spaced at distances $2a'd/d'$ and move forward with velocity

$$v = \frac{d' - d}{d'} v_0 \,. \tag{21}$$

If

$$a'd < ad'$$

the shadow rails move *forward* with velocity

$$V = \frac{a(d' - d)}{ad' - a'd} v_0 = v_0 + \frac{(a' - a)d}{ad' - a'd} v_0. \tag{22}$$

They always move in the same direction as the observer, but overtake him or fall behind according as the more distant set is more widely spaced than the nearer set or not.

If

$$a'd > ad',$$

the shadow rails are bounded by rails of the nearer set and move backwards relatively to the further set with velocity

$$\frac{(d' - d)a'd}{(a'd - ad')d'} v_0 \,.$$

The velocity relative to the rails of the *nearer* set is given, as in the first case, by equations (22).

Now let the observer approach from a great distance so that the ratio d/d' decreases from unity to zero. Two cases arise when

$$b' + b > a, \qquad \text{according as} \qquad a \gtrless a'. \tag{23}$$

In the first case it follows, by replacing a', b' in Theorem II by $a'd/d'$, $b'd/d'$ respectively, that the size of the shadow rails, viewed in the plane of the nearer set, continuously decreases as the observer approaches. Ghosts, as defined in section 2, will be observed when

$$\frac{d}{d'} < \frac{a - b}{b'} \,. \tag{24}$$

From Theorems III and IV it may be seen that the spacing of the shadow rails or ghosts, as the case may be, continually decreases whilst the ratio B/A increases towards the limit $(b/a + b'/a')$.

In the second case defined by (23) the shadow rails become infinite when $a'd = ad'$. From this point on, the effect of approaching the rails is much as in the first case.

7. The effect of angular inclination. Let the slope, the tangent of the angle of inclination, of the second set of rails relative to the first set be s and let S denote the slope of the shadow rails relative to the first set. The open spaces common to both sets are parallelograms arranged in rows. When s is small the parallelograms are long.

The results to be obtained in this section follow easily from those at the beginning of the previous section. The figure to be considered may be regarded as the space-time graph of the motion of the second set of rails relative to the first with uniform velocity s if the edge of a rail of the first set is taken as the time axis. Applying equation (19), the average slope of the shadow rails relative to the first set is given by

$$S = \frac{a}{a - a'}\, s. \tag{25}$$

8. The effect of perspective. When the planes of the two sets of rails are not parallel, the projections of the second set on the plane of the first are concurrent instead of parallel. As in the case just considered, the boundaries of the shadow bands may be smoothed out by a line joining the common points of a rail 1 of the first set and rail 1 of the second set, rail 2 of the first set and rail 2 of the second set, and so on. But in this case the line is not straight, but curved.

Take as axis of x the vanishing line of the plane of the second set when the plane of the first set is the picture plane. Take as axis of y the straight line through the vanishing point of the second set of rails, parallel to the first set. The equations of the edges of the nth rail of the first set and the n'th rail of the second set may be written respectively

$$x = 2na + c \pm b, \tag{26}$$

$$x = y(2n'a' + c' \pm b'). \tag{27}$$

The shadow bands will be more clearly defined in those parts of the plane in

which the angle between the two sets of rails is small. Such a region is that included between

$$x = \pm a\, y, \tag{28}$$

where a has some value such as $\frac{1}{2}$ or $\frac{1}{3}$.

The boundaries of the shadow bands may now be defined by writing

$$n' - n = N \tag{29}$$

and eliminating n, n' from equations (26), (27) and (29). The equation thus obtained is

$$\frac{x}{a'y} - \frac{x}{a} = 2N + C, \tag{30}$$

where

$$C = \frac{c' \pm b'}{a'} - \frac{c \pm b}{a}. \tag{31}$$

Equation (30) represents a hyperbola with asymptotes

$$a'y = a,\ x + (2N + C)\, a = 0. \tag{32}$$

The shadow bands are therefore bounded by hyperbolas with asymptotes parallel to the first set of rails and the vanishing line of the plane of the second set of rails respectively. The latter asymptote is fixed and therefore common to all the bands. It is that line for which the distance between intersections with the center lines of consecutive rails is the same for both sets.

An important exception occurs when both the two sets of rails are parallel to the vanishing line. In this case the shadow bands must reduce to shadow rails parallel to the given sets but they will no longer be uniformly spaced. For our present purpose it is sufficient to consider the case in which the observer is at a distance large compared with the shadow rails. In this case the spacing of the shadow rails in the neighborhood of any point may be calculated as if the second set appeared as uniform rails when viewed in the plane of the first set. Let the planes of the two sets of rails make angles θ, θ' with the plane passing through their join and the observer's eye. Let a plane through a rail to be considered and the observer's eye make an angle ϕ with this plane.

The distance apart of rails in the second set being $2a'$ and their apparent distance, viewed in the plane of the first set, at this point being $2a''$,

$$a'' = a' \frac{\sin\theta \sin^2(\phi + \theta')}{\sin\theta' \sin^2(\phi + \theta)} . \tag{33}$$

If $\theta' > \theta$, as ϕ increases from 0 to $\pi - \theta'$, a'' decreases from $a' \sin\theta' / \sin\theta$ to 0. Two cases therefore arise according as

$$a' \sin\theta' \gtrless a \sin\theta. \tag{34}$$

In the first case A increases with ϕ and becomes infinite when

$$a \sin\theta' \sin^2(\phi + \theta) = a' \sin\theta \sin^2(\phi + \theta'). \tag{35}$$

For greater values of ϕ, A continually decreases, as also occurs in the second of the two cases indicated by (34).

Since, by Theorem I, the direction of motion of the shadow rails relative to the larger set is the same as that of the more closely spaced set, and since in the neighborhood of the value of ϕ determined by equation (35) A is large, it follows that the effect of motion on the part of the observer in any direction but that of ϕ will be to cause the shadow rails on opposite sides of this critical position to move in opposite directions.

Further, the rails appear to converge towards or diverge from this spot according as the observer's path makes an angle greater or less than ϕ with the plane through his eye and the join of the planes of the two sets of rails.

If the direction of motion is towards the rails, the shadows will converge towards or diverge from the point approached because the relative motion of the two sets of rails changes sign at this point. But this motion is slow, because of the small relative velocity at this point, while that just considered is fast, because of the large magnification.

Finally, equation (35) may be put into the more convenient form

$$\frac{\cot\phi + \cot\theta}{\cot\phi + \cot\theta'} = \left(\frac{a' \sin\theta'}{a \sin\theta}\right)^{\frac{1}{2}} . \tag{36}$$

If we put

$$a = a', \qquad \theta' = \theta + \tfrac{1}{2}\pi, \qquad x = \cot\phi, \qquad c^2 = \cot\theta,$$

we obtain

$$x = \frac{c^3 + 1}{c^2 - c} .$$

The value of x is positive only when $c > 1$, that is when $\theta < 45°$. A short calculation shows that the minimum value of x, and therefore the maximum value of ϕ, occurs when c lies near $2{\cdot}2$, making θ and ϕ each about 12°.

The numerical values have been chosen to represent the case of two uniform sets of rails meeting at right angles. Observation confirms these results, though this is a case in which the substitution of round or square rails for flat ones makes a considerable difference. Owing to the large magnification at the critical angle, any slight deviation of the plane of either set of rails from the vertical causes an appreciable curvature in the shadow rails.

HAVERFORD COLLEGE,
HAVERFORD, PA.